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ORIGINAL PAPER

Approach on Tsallis statistical interpretation of hydrogen-atom by adopting the generalized radial distribution function

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Abstract This paper revisits the statistical interpretation of the hydrogen atom within the framework of Tsallis Statistical Mechanics in the Canonical Ensemble. The convergence of the partition function does not exhibit for all the temperatures, while the well-known $T \rightarrow T'$ transformation method of Tsallis Statistics fails, since non-monotonicity is observed between the ordinary temperature, T, and the auxiliary one, T'. Here we re-examine the inconsistency of $T \rightarrow T'$ transformation method, in the case where the partition function converges for all the temperatures, by considering the generalized radial distribution function. We find that both the transformation method inconsistency and the partition function divergence can be recovered for all the temperatures, if the hydrogen atom is restricted within a critical radius $R_c \leq 4.832$ bohr, while Tsallis entropic index values are given by $q\ (R_c) \in \left[q_c \cong 0.664, q^* = \frac{7}{9}\right]$.

Keywords Hydrogen-atom · Generalized radial distribution function · Tsallis Statistics

1 Introduction

In the last two decades, the Boltzmann–Gibbs statistical thermodynamics was successfully generalized to the non-extensive thermostatistical formulation, proposed by Tsallis [1]. Indeed, many physical systems that cannot by explained correctly in the classical statistical description, found their convincing description within the framework of non-extensive Statistics [2,3]. One of the significant applications of Tsallis Statistics concerns the partition function for the discrete energy levels of the hydrogen-atom

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in the canonical ensemble. In particular, Lucena, da Silva and Tsallis calculated the specific heat of a hydrogen-atom in the canonical ensemble [4], where the obtained results depend on the selected values of $q < q^*$, where the critical value $q^* = \frac{7}{9}$ was discussed in [3,5].

The relevant pioneer publication of Tsallis [1] suggests the following generalized entropy formulation

$$S_q = \frac{1}{q-1}\left(1 - \sum_{k=1}^{W} p_k^q\right), \tag{1}$$

where the Boltzmannian formulation can be uniformly included in that of Tsallis for $q \to 1$ [1–3].

The concept of the Canonical Ensemble involves the extremalization of S_q with the constraints of the norm $\sum_{k=1}^{W} p_k = 1$ and of the normalized q-expectation value of energy $\sum_{k=1}^{W} P_k \varepsilon_k = U_q$, expressed in terms of the dual, escort probabilities $P_k\left(\{p_i\}_{i=1}^{W}; q\right) \equiv p_k^q / \sum_{k=1}^{W} p_k^q$ (inversed as $p_k\left(\{P_i\}_{i=1}^{W}; q\right) \equiv P_k^{1/q} / \sum_{k=1}^{W} P_k{}^{1/q}$), yielding the following thermal equilibrium probability distribution of the energy spectrum $\{\varepsilon_i\}_{i=1}^{W}$ as [1–4]:

$$p_k\left(\{\varepsilon_i\}_{i=1}^{W}; \beta; q\right) = g_k \frac{1}{Z_q}\left[1 - (1-q)\,\beta' \varepsilon_k\right]^{\frac{1}{1-q}}, \tag{2}$$

with the partition function given by

$$Z_q = \sum_{k=1}^{W} g_k \left[1 - (1-q)\,\beta' \varepsilon_k\right]^{\frac{1}{1-q}}, \tag{3}$$

where

$$\beta' \equiv \frac{\beta}{\sum_{k=1}^{W} p_k^q + (1-q)\,\beta U_q} \equiv \frac{1}{k_B T'}, \tag{4}$$

$$\beta \equiv \frac{1}{k_B T}, \tag{5}$$

and $\{g_i\}_{i=1}^{W}$ are the relevant degeneracy spectrum of the energy states $\{\varepsilon_i\}_{i=1}^{W}$.

Speaking more precisely, the quantity within the outer brackets of Eqs. 2 and 3 has to be nonnegative. Therefore, we usually rewrite Eqs. 2 and 3 in terms of the modified brackets $[\cdot]_+$, having the following meaning: $[u]_+ = u$, for $u \geq 0$, and $[u]_+ = 0$, for $u \leq 0$, expressing the so-called "cut-off condition" of Tsallis [2,3]. Throughout, however, we will ignore this symbolism for simplicity, while we will retrieve it when is necessary.

By introducing the q-expectation value of energy $\sum_{k=1}^{W} P_k \varepsilon_k = U_q$, instead of the classical one $\sum_{k=1}^{W} p_k \varepsilon_k = U$ [2,3,5,6], Tsallis succeeded, among others, in recovering the additivity relation of the expectation values of energy (internal energies) of two subsystems, and the invariance of the probability distribution, with respect to the energy ground level arbitrary definition.

On the other hand, the problem of the implicit expression of the probabilities $\{p_i\}_{i=1}^{W}$ is overcome through the transformation (4), where β' should be handled as intermediate parameter which is used only for computational sake. In particular, [7] summarized the following computational steps, in order the probability implicitness to be avoid: (i) Computation of the quantities $y_k \equiv 1 - (1-q)\beta'\varepsilon_k, \forall k = 1, \ldots, W$. (ii) Cut-off condition: If $y_k < 0$ then set $y_k = 0$. (iii) Compute $Z_q = \sum_{k=1}^{W} y_k^{\frac{1}{1-q}}$. (iv) Derivation of the connecting thermodynamical quantities, such as the internal energy, in terms of the intermediate parameter β'. (v) Retrieving of β, through the expression $\beta = \beta\left(\beta'\right)$, given in Eq. 4.

In their recently published paper [8], Barati and Moradi claimed the inconsistency of the above described the transformation $\beta \to \beta'$ method, when it is applied to a hydrogen-atom in the canonical ensemble. As Tsallis et al. pointed out [6], after the elimination of the possible quantum originated spurious oscillations which disappear at high temperatures, one must obtain the re-normalized temperature T, as a monotonically increasing function of the intermediate one T'. However, Barati and Moradi observed that for the accepted values of q (namely, $q < \frac{7}{9}$), the re-normalized temperature T does not behave as a monotonically increasing function of the intermediate temperature T', for the allowable range of q for the hydrogen-atom.

On the other hand, the partition function for the discrete energy levels of the hydrogen-atom is thought to be given by Eq. 3, where $\varepsilon_k = R_H\left(1 - 1/k^2\right)$ (R_H is the Rydberg constant), suitably modified for the ground state energy being at the origin, namely,

$$Z_q\left(t'\right) = \sum_{k=1}^{\infty} 2k^2 \left[1 - \frac{1-q}{t'}\left(1 - \frac{1}{k^2}\right)\right]^{\frac{1}{1-q}}, \tag{6}$$

where the reduced temperature t' (and t) is defined by $1/t' \equiv \beta' R_H$ (and $1/t \equiv \beta R_H$). Unfortunately, as it is well known, this partition function suffers from divergence for the Boltzmannian case $q \to 1$, for all the values of temperatures. (Strictly mathematically speaking, this is caused by the term implied by the degeneracy $g_k = 2k^2$, since for large k^*, we have $Z \approx const.(t; k^*) + \sum_{k=k^*+1}^{\infty} k^2$, which obviously diverges.) However, in general, $Z_q\left(t'\right)$ diverges for $t' > 1-q$, while converges in the t'-interval $D_{t',Conv} = (0, 1-q] \subset \Re^+$.

Hence, we conclude in the following remarks:

– First, the results of Barati and Moradi are undoubtedly correct, since they are easily replicated and verified. In summary, these are the following: The partition function for the discrete energy levels of the hydrogen-atom does converge only for temperature values $t' \in D_{t',Conv}$. On the other hand, the subinterval of the consistency values of t', namely, $D_{t',Consist} = \left(0, t'^*\right) \subset D_{t',Conv} \subset \Re^+$, where $t'^* < 1-q$, is included within the converging interval $D_{t',Conv}$, rather than coinciding with it. Hence, there is an interval of inconsistency, namely, $D_{t',Inconsist} = \left(t'^*, 1-q\right) \neq \emptyset$.

– However, by proposing the two alternative methods, the iterative and the $\beta \rightarrow \beta'$ transformation methods, Tsallis et al. [6] remarked that their equivalence is an outcome of two preconditions. The first one, as we mentioned, is constituted by the monotonically increasing function $t\left(t'\right)$(or by $\beta\left(\beta'\right)$). Tsallis et al. [6] found that there is a lower limit for entropic indices q_c for which this is true. (The critical value q_c depends on the particular system. E.g. for the quantum harmonic oscillator $q_c \cong 0.56$ [7].) The second precondition is not discussed by [6], because it is trivial: The partition function (and any relevant summations) has to converge for all temperature values, as it is naturally expected. (None invalid temperature value is known, except of the finite number of temperature values where we have phase transitions.) If the second precondition is not fulfilled, then it is not self-evident that there exists any entropic index for which $t\left(t'\right)$ is monotonically increasing function.

Therefore, one first purpose of this paper is to investigate the case where the divergence is totally recovered, namely $D_{t',Conv} = \Re^{+}$. One conceptual and consequent way of avoiding the apparent divergence of the partition function in Boltzmann–Gibbs Statistics involves considering the so-called generalized radial distribution function. We will show that this also holds in the case of Tsallis Statistics.

Thereafter, we examine whether the inconsistency still exists or is recovered, when the divergence is totally recovered adopting the generalized radial distribution function, i.e., $D_{t',Consist} = \Re^{+}$, when $D_{t',Conv} = \Re^{+}$. We will show that both can be recovered, if the hydrogen atom is restricted within a critical radius. In this case, indeed, we find $D_{t',Conv} = D_{t',Consist} = \Re^{+}$.

2 The generalized radial distribution function

As it was mentioned, the Boltzmannian partition function, given by the limit $q \rightarrow 1$ of the Tsallis-like partition function, given in Eq. 6.

$$Z_B\left(t' = t\right) = \sum_{k=1}^{\infty} 2k^2 e^{-\frac{1}{t}\left(1-\frac{1}{k^2}\right)}, \tag{7}$$

suffers from divergence for any temperature value.

However, for the hydrogen-atom imposed in a Riemannian space of constant positive Gaussian curvature $1/R$, Schrödinger [9] showed that the energy eigenvalues are given by

$$\varepsilon_k = R_H\left(1 - \frac{1}{k^2} + \frac{k^2-1}{R^2}\right), \tag{8}$$

where the dimensionless radius R counts the Bohr-radii. Hence, the divergence of the relevant Boltzmannian partition function recovers in

$$Z_{B,R}\left(t\right) = \sum_{k=1}^{\infty} 2k^2 e^{-\frac{1}{t}\left(1-\frac{1}{k^2}+\frac{k^2-1}{R^2}\right)}. \tag{9}$$

We stress that the partition function converges for any finite value of R, namely, $Z_{B,R}(t) < +\infty$. This is apparent because as it well known, for any polynomial $P(x;d)$ of degree d, a critical value x^* can always be found, so that $\forall x > x^*$ the exponential function e^x growths faster than $P(x;d)$. Hence, by setting $P(x;d) \equiv x^2$, we have that

$$\sum_{k=k^*}^{\infty} \frac{k^2}{e^{\frac{k^2}{tR^2}}} < \sum_{k=k^*}^{\infty} \frac{k^2}{\left(\frac{k^2}{tR^2}\right)^2} = t^2 R^4 \sum_{k=k^*}^{\infty} \frac{1}{k^2} < +\infty.$$

Hence,

$$Z_{B,R}(t) = \sum_{k=1}^{\infty} 2k^2 e^{-\frac{1}{t}\left(1-\frac{1}{k^2}+\frac{k^2-1}{R^2}\right)} = \sum_{k=1}^{k^*} 2k^2 e^{-\frac{1}{t}\left(1-\frac{1}{k^2}+\frac{k^2-1}{R^2}\right)} + \sum_{k=k^*}^{\infty} 2k^2 e^{-\frac{1}{t}\left(1-\frac{1}{k^2}+\frac{k^2-1}{R^2}\right)},$$

where

$$\begin{aligned} \sum_{k=k^*}^{\infty} 2k^2 e^{-\frac{1}{t}\left(1-\frac{1}{k^2}+\frac{k^2-1}{R^2}\right)} &= 2e^{-\frac{1}{t}\left(1-\frac{1}{R^2}\right)} \sum_{k=k^*}^{\infty} k^2 e^{-\frac{k^2}{tR^2}} e^{\frac{1}{tk^2}} \\ &< 2e^{-\frac{1}{t}\left(1-\frac{1}{R^2}\right)} e^{\frac{1}{tk^{*2}}} \sum_{k=k^*}^{\infty} k^2 e^{-\frac{k^2}{tR^2}} \\ &< 2e^{-\frac{1}{t}\left(1-\frac{1}{R^2}\right)} e^{\frac{1}{tk^{*2}}} t^2 R^4 \sum_{k=k^*}^{\infty} \frac{1}{k^2} < +\infty. \end{aligned}$$

so that

$$Z_{B,R}(t) < \sum_{k=1}^{k^*} 2k^2 e^{-\frac{1}{t}\left(1-\frac{1}{k^2}+\frac{k^2-1}{R^2}\right)} + 2e^{-\frac{1}{t}\left(1-\frac{1}{R^2}\right)} e^{\frac{1}{tk^{*2}}} t^2 R^4 \sum_{k=k^*}^{\infty} \frac{1}{k^2} < +\infty.$$

Of course, and as Blinder [10] remarked, we do not claim that the actual curvature of space-time is capable for having any perceptible effects on atomic structures. However, the Riemannian curvature is an effectual metaphor for representing the influence of generalized radial distribution function defined in the 3-dimensional Euclidean space occupied by the atom.

The generalized radial distribution function $D_n(r)$ was introduced to characterize the spherically symmetrical function resulting from a summation over all the angular momentum states for a given energy [10–12]. In such a case, the contribution of the discrete states of the hydrogen-atom leads to a partition function of the form $Z_{B,R}(t)$, while, however, the radius R is related to the Laboratory-size volume V, namely, $V = 2\pi^2 R^3$ (in Reimannian space).

Here, we are interested in integrating into the vicinity of the atom, namely, R is reduced to the atomic scales, that is of the order of several Bohr radii. On the other hand, within the framework of Tsallis Statistical Mechanics, the relevant partition function is given by

$$Z_{q,R}\left(t'\right) = \sum_{k=1}^{\infty} 2k^2 \left[1 - \frac{1-q}{t'}\left(1 - \frac{1}{k^2} + \frac{k^2-1}{R^2}\right)\right]^{\frac{1}{1-q}}. \qquad (10)$$

Throughout, we are approach the problem, considering the restoring term $\left(k^2-1\right)/R^2$ in energies, as it is given in Eq. 8.

3 Results

3.1 Partition function convergence

In Fig. 1a we examine the "partial partition function", given by

$$Z_{q,R}^{N}\left(t', N\right) = \sum_{k=1}^{N} 2k^2 \left[1 - \frac{1-q}{t'}\left(1 - \frac{1}{k^2} + \frac{k^2-1}{R^2}\right)\right]^{\frac{1}{1-q}}, \qquad (11)$$

where we numerically verify its convergence as $N \to \infty$, for $q = 0.7$, $t' = 0.5 > 1-q = 0.3$, and $R = 10^2, 10^3, 10^4, 10^5$, and 10^6. The convergence is also shown in Fig. 1b for $q = 0.7$, $R = 10^4$ and $t' = 0.4, 0.5, 1, 2$, and 5.

Moreover, the convergence of $Z_{q,R}\left(t'\right)$, as given by Eq. 10, can be also analytically shown, as follows.

The partition function for $R \to \infty$, i.e., $Z_q\left(t'\right)$, as given by Eq. 6, has the quantity within the outer brackets, i.e., $1 - \frac{1-q}{t'}\left(1 - \frac{1}{k^2}\right)$, being always positive for

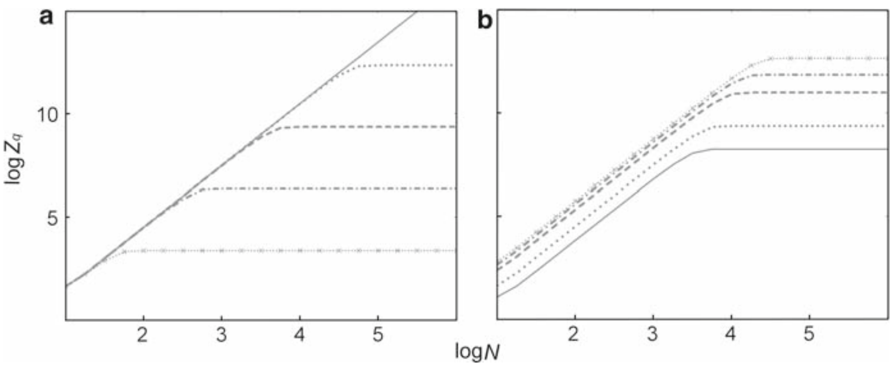

Fig. 1 The convergence of the partial Tsallis-like partition function, given in Eq. 11 **a** for $q = 0.7$, $t' = 0.5 > 1-q = 0.3$, and $R = 10^2$ (*cross-line*), 10^3 (*dashdot-line*), 10^4 (*dash-line*), 10^5 (*dot-line*), 10^6 (*solid-line*), and **b** for $q = 0.7$, $R = 10^4$ and $t' = 0.4$ (*solid-line*), 0.5 (*dot-line*), 1 (*dash-line*), 2 (*dashdot-line*), 5 (*cross-line*)

$t' > 1-q$, and lying in the interval $0 < 1-\frac{1-q}{t'}\left(1-\frac{1}{k^2}\right) \leq 1$. Indeed, we have $1-\frac{1-q}{t'}\left(1-\frac{1}{k^2}\right) \leq 1-\frac{1-q}{t'}(1-1) = 1$ and $1-\frac{1-q}{t'}\left(1-\frac{1}{k^2}\right) \geq 1-\frac{1-q}{t'} > 0$. However, for $t' \leq 1-q$ the quantity $1-\frac{1-q}{t'}\left(1-\frac{1}{k^2}\right)$ is not always positive. In particular, it is positive for sufficiently small values of k, while as k increases it becomes smaller and for $k > k^*$, where $k^* = \left(1-\frac{t'}{1-q}\right)^{-1/2}$, it is negative. (Obviously, the value of k^* can be readily found by setting $1-\frac{1-q}{t'}\left(1-\frac{1}{k^{*2}}\right) = 0$. Moreover, we apparently take into account only the integer part, i.e., $k^* = \text{Integer}\left\{\left(1-\frac{t'}{1-q}\right)^{-1/2}\right\}$, but we ignore this symbolism for simplicity.). Thus, the maximum value of k is k^*, due to the Tsallis cut-off condition. This is the reason of the convergence of $Z_q(t')$ for $t' \leq 1-q$.

Similarly in the case of finite value of R, the partition function $Z_{q,R}(t')$, as given by Eq. 10, does converge for the same reason, namely, due to the Tsallis cut-off condition, that implies a maximum value of k, that is k_R^*. Indeed, the quantity within the outer brackets of Eq. 10 is $1-\frac{1-q}{t'}\left(1-\frac{1}{k^2}+\frac{k^2-1}{R^2}\right)$, is monotonically decreasing as k increases, being equal to zero for sufficiently large values of k. (The function $f(x) = 1-\frac{1}{x^2}+\frac{x^2-1}{R^2}$ is monotonically increasing for any $x \geq 1$, since $f'(x) = 2\frac{1}{x^3}+2\frac{x}{R^2}$.) Also, by setting $1-\frac{1-q}{t'}\left(1-\frac{1}{k^2}+\frac{k^2-1}{R^2}\right) = 0$, we readily find that $k_R^* = \sqrt{b+\sqrt{b^2+R^2}}$, where $2b \equiv \left(\frac{t'}{1-q}-1\right)R^2+1$.

We observe that for any finite value of R, k_R^* is well defined, independently of the value of $\frac{t'}{1-q}$. Hence, the series involved in $Z_{q,R}(t')$ ceases, due to the cut-off condition, independently of the value of $\frac{t'}{1-q}$. Therefore, $Z_{q,R}(t')$ converges for any finite value of R.

On the other hand, however, for $R \to \infty$, k_R^* has the asymptotical behavior $k_R^* \to R\left(\frac{t'}{1-q}-1\right)^{1/2} \to +\infty$ for $\frac{t'}{1-q} > 1$, and $k_R^* \to k^* = \left(1-\frac{t'}{1-q}\right)^{-1/2} < +\infty$ for $\frac{t'}{1-q} < 1$, and $k_R^* \to \sqrt{R} \to +\infty$ for $\frac{t'}{1-q} = 1$. Namely, for $\frac{t'}{1-q} \geq 1$ the series involved in $Z_q(t')$ does not cease, while it does cease for $\frac{t'}{1-q} < 1$, due to the cut-off condition. Therefore, $Z_q(t')$ converges for $R \to +\infty$, only for $\frac{t'}{1-q} < 1$.

Hence, for $R < +\infty$, the interval of the temperature values for which the convergence of the partition function holds is $D_{t',Conv} = \Re^+$. This result is remarkable since partition function has to converge for all the temperature values, as it is naturally expected, while none invalid temperature value has to emerge through the statistical interpretation.

However, one may lodge the following objection: Once we utilize the generalized radial distribution function, the Boltzmann–Gibbs Statistics is sufficient for the statistical interpretation of thermodynamics regarding the hydrogen atom, since the partition function converges. Then, by means of what insights the adoption of Tsallis Statistics is necessary to be applied? Nevertheless, the key role of Tsallis generalized statistical interpretation of thermodynamics does not exhaust by far all its potentialities by describing systems that Boltzmann–Gibbs Statistics cannot be applicable. (A glance to the relevant literature reveals numerous examples of systems that Tsallis Statistics found to fit, yielding a better description than Boltzmann–Gibbs Statistics does.)

3.2 The monotonicity of $t\left(t'\right)$

Now, let examine the interval of the temperature values for which $t' = t'(t)$ is monotonically increasing function.

In Fig. 2a we observe the existence of inconsistency intervals, characterizing the non-monotonicity of $t\left(t'\right)$, for the case of the entropic indices $q = 0.3, 0.5, 0.7, 0.73$, and 0.76, while $R \to +\infty$. However, in the respective Fig. 2b where $R = 3.276$, we observe that the inconsistency recovers for $q = 0.7, 0.73$, and 0.76, while it still remains for $q = 0.3, 0.5$.

In particular, there is a critical value $q_c \cong 0.664$, for which, $\forall q \in [q_c \cong 0.664, q^* = \frac{7}{9}]$ we can find a critical radius $R_c\,(q)$: $\forall R \le R_c\,(q)$ the inconsistency recovers for all the temperature values, namely, $D_{t',Consist} = \Re^+$. The following Table 1 reads some characteristic values of $R_c\,(q)$:

Then, the expression of $R_c\,(q)$ can be satisfactorily approximated by the linear form

$$R_c\,(q) \cong 2.741 + 18.7\,(q - q_c)\,0. \tag{12}$$

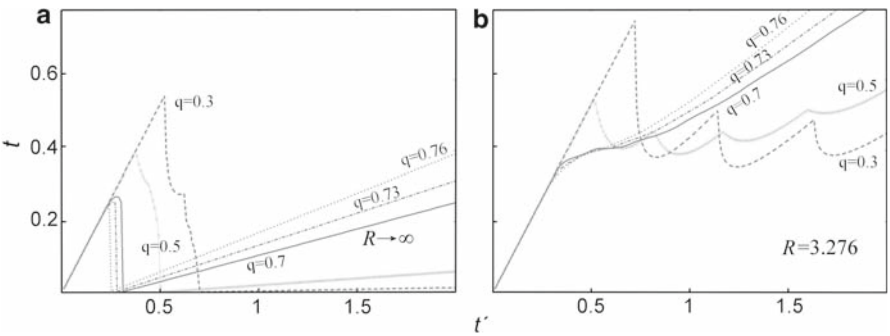

Fig. 2 The function $t\left(t'\right)$ is depicted for the entropic indices $q = 0.3$ (*dash-line*), 0.5 (*cross-line*), 0.7 (*solid-line*), 0.73 (*dash-dotline*), and 0.76 (*dot-line*), and for **a** $R \to +\infty$, **b** $R = 3.276$. We observe that the inconsistency intervals recover for $q = 0.7, 0.73$, and 0.76. (Compare with the table values)

Table 1 Some characteristic values of $R_c(q)$ for which the inconsistency recovers $\forall R \le R_c(q)$ and $\forall q \in \left[q_c \cong 0.664, q^* = \frac{7}{9}\right]$

q	$R_c\,(q)$
$q_c \cong 0.664$	2.741
0.67	2.826
0.7	3.276
0.73	3.935
0.76	4.511
$q^* \cong 0.777$	4.832

Therefore, for all $q \in \left[q_c \cong 0.664, q^* = \frac{7}{9}\right]$ and $R \leq R_c(q)$, where $R_{c,Min} \leq R_c(q) \leq R_{c,Max}$, with $R_{c,Min} = 2.741$, $R_{c,Max} = 4.832$, we conclude in $D_{t',Consist} = D_{t',Conv} = \Re^+$, and the inconsistency of the transformation method of Tsallis Statistics regarding the hydrogen atom recovers.

Note that the inverse of Eq. 12, namely,

$$q(R_c) \cong q^* + \frac{1}{18.7}\left(R_c - R_{c,Max}\right), \tag{13}$$

reads that we can attain to $D_{t',Consist} = D_{t',Conv} = \Re^+$, for all $R \leq R_c$: $R_c \in \left[R_{c,Min}, R_{c,Max}\right]$ and $q = q(R_c) \in \left[q_c, q^*\right]$.

We stress the fact that, in the past, the generalized radial distribution function was utilized by being integrated in a radius R, that is of the laboratory order of dimensions. However, it is not exclusionary to be integrated in a radius R of the order of several atom radii. Indeed, limitations on the electron radial distance from the nucleus cannot exclusively provided by the macroscopic box boundaries, but also from the presence of the other hydrogen atoms. This consideration signifies the meaning of the mean free path assigned between the atoms.

Finally, we remark that there is no need for the interval of the R-values, $D_{R,Conv}$, for which the convergence of $Z_{q,R}$ exhibits, i.e., $D_{R,Conv} = (0, +\infty)$, to coincide with the interval of the R-values, $D_{R,Consist}$, for which the inconsistency recovers, i.e., $D_{R,Consist} = [0, R_c(q)]$ (in the same manner that the respective intervals of the q-values shall differ).

4 Conclusions

This paper revisited the statistical interpretation of the hydrogen atom. Within the framework of Tsallis Statistical Mechanics, the case where the divergence of the partition function is totally recovered by adopting the generalized radial distribution function, was thoroughly examined.

Thereafter, we examined whether the inconsistency still exists or is recovered, when the divergence is totally recovered by adopting the generalized radial distribution function. We found that both can be recovered: The validity of the $T \rightarrow T'$ transformation method was ensured as soon as the partition function converge for all the temperature values.

The necessity of the partition function to be converging for all the temperature values, as it is naturally expected, while none invalid temperature value has to be emerging through the statistical interpretation, was stressed out.

In particular, we sought for the critical values of the entropic indices for which the monotonicity of $t\left(t'\right)$(or of $\beta\left(\beta'\right)$) holds. (The $T \rightarrow T'$ transformation method is ensured when this monotonicity holds. E.g., in the case of the quantum harmonic oscillator, where the partition function converges for all the temperature values, $t\left(t'\right)$ is monotonically increasing function for $q > q_c \cong 0.56$ [6]). Moreover, the generalized radial distribution function was integrated in a radius R, being of the order of several atom radii. As it was claimed, limitations on the electron radial distance from the nucleus cannot exclusively provided by the macroscopic box boundaries, but also from the presence of the other hydrogen atoms.

Finally, we found that for all the entropic indices $q \in \left[q_c \cong 0.664, q^* = \frac{7}{9}\right]$ and the radii $R \leq R_c(q)$, where $2.741 \leq R_c(q) \leq 4.832$, the inconsistency of Tsallis Statistics (regarding the hydrogen atom [8]) is totally recovered, in similar to the convergence of the partition function Z_q, that is for all the temperature values.

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